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Stationarity-conservation laws for certain linear fractional differential equations

Małgorzata Klimek

Institute of Mathematics and Computer Science, Technical University of Częstochowa, ul. Dąbrowskiego 73, 42-200 Częstochowa, Poland

E-mail: klimek@matinf.pcz.czest.pl

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Abstract

The Leibniz rule for the fractional Riemann–Liouville derivative is studied in the algebra of functions defined by Laplace convolution. This algebra and the derived Leibniz rule is used in construction of an explicit form of stationary-conserved currents for linear fractional differential equations. The examples of fractional diffusion in $1 + 1$ and fractional diffusion in $d + 1$ dimensions are discussed in detail. The results are generalized to the mixed fractional–differential and mixed sequential fractional–differential systems for which the stationarity-conservation laws are obtained. The derived currents are used in construction of stationary nonlocal charges.

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1. Introduction

In this paper we shall study the properties of the fractional–differential equations together with the mixed models containing both fractional and standard classical derivatives.

The fractional analysis describing the fractional integrals and derivatives is covered extensively in the literature (see e.g. the monographs [1–4] and references given therein). Recently these operators have found application in various areas of physics. Let us start with the fractional mechanics describing the nonconservative systems developed by Riewe [5, 6], who also shows the possible connection between the fractional formalism and a problem of classical frictional force proportional to velocity.

The fractional operators emerge also as the infinitesimal generators of coarse-grained macroscopic time evolutions [7–11] and determine fractional diffusion processes [4, 12–16].

The phenomenological approach to derivation of the stress–strain relationships, which tends to proper description of the rheological properties of wide classes of materials, leads to rheological constitutive equations with fractional derivatives [17].

The next domain is the path-integral formulation of classical boundary problems with fractal boundaries used in polymer science. These models can be rewritten in the form of fractional differential equations. The order of the fractional operator is given by the geometry of the boundary, the space in which the boundaries are embedded and the type of random walk process [17, 18].

One should also mention the description of wandering processes given by the fractional Fokker–Planck–Kolmogorov equation in fractal spacetime [19–22], the fractional generalization of the Klein–Kramers equation, which yields the fractional Raleigh and Fokker–Planck models [23–25], and the fractional equation describing the end-to-end distribution of a stable random walk where the fractional power of the standard Laplace operator is used [18].

Finally the fractional operators appear also in field theory, where recently the roots of the wave operator were investigated by Zavada [26, 27]. In his paper he shows that the Dirac operator is the only one of them which can be realized using the standard derivatives. When the root of order different from $\frac{1}{2}$ is considered we obtain the fractional differential equation.

Most of the above examples are linear equations with constant coefficients of mixed type—containing both fractional and standard derivatives. As is well known in classical field theory the conservation laws for linear differential systems can be derived using the Takahashi–Umezawa method [28]. This procedure has been extended to discrete and noncommutative models [29–31].

Our aim is to show that a similar procedure can be applied to fractional equations in the convolution algebra of functions in order to obtain the stationarity-conservation laws, which are analogues of conservation equations known for models from classical differential calculus. The explicitly derived stationary-conserved currents are nonlocal expressions with respect to this part of space, for which the fractional derivatives appear in the initial equation. This phenomenon is connected with the nonlocality of fractional operators as well as with the convolution algebra of functions which we introduce to simplify the Leibniz rule in the fractional differential calculus.

Some of the derived nonlocal currents yield the stationary charges which in turn can be converted into nonlocal conserved charges. In this paper we discuss this procedure on some examples and then for the general case of mixed fractional and differential equations. These nonlocal integro-differential equations obey a new type of conservation law, which we call the stationarity-conservation law.

In section 2 we review briefly the properties of Riemann–Liouville fractional integrals and derivatives and show that the Leibniz rule is simplified in the algebra of convolution. The new Leibniz rule produces strict requirements concerning the behaviour of the functions in the neighbourhood of zero. The next section contains the detailed discussion of the derivation of the stationarity-conservation law for two examples of fractional diffusion: in $1 + 1$ and $d + 1$ dimensions. It is explicitly proven that the asymptotic properties of the solutions for the diffusion equation in $1 + 1$ dimensions allow construction of stationary currents and stationary charges. Then the currents and charges are converted via convolution to conserved currents and charges, which are stationary in a strict sense—that means true constant functions.

The final section includes the general fractional–differential model as well as the sequential fractional–differential one. We show the explicit construction of stationary currents and the derivation of the stationarity-conservation laws and close the section with discussion of the possible stationary and conserved charges.

2. Properties of fractional integrals and derivatives

2.1. Riemann–Liouville fractional integral

Let us recall the definition of the Riemann–Liouville fractional integral [1–3] used widely in the literature dealing with fractional calculus:

Definition 2.1. *Let* $\operatorname{Re}\nu > 0$ *and let* f *be piecewise continuous on* $(0, +\infty)$ *and integrable on any finite subinterval of* $[0, +\infty)$. *Then for* $t > 0$

$$D_t^{-\nu} f(t) := \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} f(s)\,\mathrm{d}s \tag{1}$$

is the Riemann–Liouville fractional integral of f *of order* ν.

We notice that the above definition includes the operation of Laplace convolution; namely, it can be written as

$$D_t^{-\nu} f(t) = \Phi_{-\nu} * f(t) = f * \Phi_{-\nu}(t) \tag{2}$$

where we have denoted $\Phi_{-\nu}(t) = \frac{1}{\Gamma(\nu)} t^{\nu-1}$.

Now we are interested in the properties of the fractional integral connected with the composition of the integrals with respect to the same coordinate. The answer is the generalization of the Dirichlet integral formula for a continuous function, which is called in fractional calculus the composition rule [1–3]:

$$D^{-\nu} D^{-\mu} f(t) = D^{-(\nu+\mu)} f(t) = D^{-\mu} D^{-\nu} f(t) \tag{3}$$

for $\operatorname{Re}\mu, \operatorname{Re}\nu > 0$ and for any function f piecewise continuous on $[0, +\infty)$.

Let us now present the known forms of Leibniz's rule for integral (1):

$$D^{-\nu}(f \cdot g) = \sum_{j=0}^{\infty} \binom{-\nu}{j} D^{-\nu-j} f \cdot D^j g \tag{4}$$

where f and g are real analytic functions on $[0, +\infty)$. This rule was generalized by Osler [1, 32–34], who obtained the following forms of Leibniz's rule:

$$D^{-\nu}(f \cdot g) = \sum_{j=-\infty}^{+\infty} \frac{\Gamma(-\nu+1)}{\Gamma(-\nu-\gamma-j+1)\Gamma(\gamma+j+1)} D^{-\nu-\gamma-j} f \cdot D^{\gamma+j} g \tag{5}$$

$$D^{-\nu}(f \cdot g) = \int_{-\infty}^{+\infty} \frac{\Gamma(-\nu+1)}{\Gamma(-\nu-\gamma-\lambda+1)\Gamma(\gamma+\lambda+1)} D^{-\nu-\gamma-\lambda} f \cdot D^{\gamma+\lambda} g\,\mathrm{d}\lambda \tag{6}$$

where γ is an arbitrary complex number.

We shall not discuss the convergence of the series in (5) and of the improper integral in (6). Let us notice however that when the algebra of functions is defined by standard point-wise multiplication as in the above formulae all versions of Leibniz's rule are very complicated.

Thus we propose to investigate the algebra of functions with multiplication defined via Laplace convolution:

$$f * g(t) := \int_0^t f(t-s) g(s)\,\mathrm{d}s. \tag{7}$$

As is well known this multiplication is associative and commutative. The neutral element is the Dirac δ-function. Let us prove the following Leibniz rule for fractional integral (1) and multiplication defined by (7):

$$D^{-\nu}(f * g) = (D^{-(\nu-\gamma)} f) * D^{-\gamma} g \tag{8}$$

where Re $\nu > 0$ and γ a complex number fulfilling inequality Re $(\nu - \gamma) \geqslant 0$.

The new Leibniz rule is implied by the composition rule (3) and properties of convolution, which defines the fractional integral (2) and algebra of functions (7):

$$D^{-\nu}(f * g) = D^{-\gamma-(\nu-\gamma)}(f * g) = D^{-\gamma} D^{-(\nu-\gamma)}(f * g) = (((f * g) * \Phi_{-(\nu-\gamma)}) * \Phi_{-\gamma})$$
$$= (f * \Phi_{-(\nu-\gamma)}) * (g * \Phi_{-\gamma}) = D^{-(\nu-\gamma)} f * D^{-\gamma} g \tag{9}$$

where Re $\nu > 0$ and Re $(\nu - \gamma) \geqslant 0$.

The derived formula (8) is similar to the multiplicity properties of the transformation operators in the discrete [29] and noncommutative [30,31] differential multidimensional calculi for standard product of functions:

$$\zeta^i_j(f \cdot g) = (\zeta^i_k f) \cdot (\zeta^k_j g). \tag{10}$$

The multiplicity property of the fractional calculus (8) leads to the following redefinition of the integral of order ν:

$$\mathcal{D}^{-\nu}_t f(t) := (D^{-\nu}_t - 1) f(t) = f * (\Phi_{-\nu} - \delta)(t). \tag{11}$$

The new operator $\mathcal{D}$ obeys the following Leibniz rule in the algebra defined by convolution product (7):

$$\mathcal{D}^{-\nu}_t(f * g) = (\zeta^{-\gamma} f) * \mathcal{D}^{-(\nu-\gamma)}_t g + (\mathcal{D}^{-\gamma}_t f) * g \tag{12}$$

or its symmetric form

$$\mathcal{D}^{-\nu}_t(f * g) = f * \mathcal{D}^{-\gamma}_t g + (\mathcal{D}^{-(\nu-\gamma)}_t f) * \zeta^{-\gamma} g \tag{13}$$

where for given ν the γ is a complex number fulfilling conditions Re $\gamma > 0$, Re $(\nu - \gamma) \geqslant 0$. As we have noticed the analogy between the action of the fractional operator D in the algebra of convolution product (7) and the transformation operator ζ in the discrete and noncommutative algebra (10) we shall use in the following the notation ζ for the 'old' fractional integral (1):

$$\zeta^{-\alpha} \equiv D^{-\alpha}_t. \tag{14}$$

The above Leibniz rules are implied by the properties of the convolution and the composition rule (3). If ν and γ fulfil the above restrictions we obtain

$$\mathcal{D}^{-\nu}_t(f * g) = f * g * (\Phi_{-\nu} - \delta) = f * g * (\Phi_{-\gamma-(\nu-\gamma)} \pm \Phi_{-\gamma} - \delta)$$
$$= f * g * \Phi_{-\gamma} * (\Phi_{-(\nu-\gamma)} - \delta) + f * g * (\Phi_{-\gamma} - \delta)$$
$$= (f * \Phi_{-\gamma}) * g * (\Phi_{-(\nu-\gamma)} - \delta) + f * (\Phi_{-\gamma} - \delta) * g$$
$$= (\zeta^{-\gamma} f) * \mathcal{D}^{-(\nu-\gamma)}_t g + (\mathcal{D}^{-\gamma}_t f) * g. \tag{15}$$

The proof of the symmetric form of the Leibniz rule (13) is analogous.

2.2. Riemann–Liouville fractional derivative

The operator known as the Riemann–Liouville fractional derivative [1–3] is defined using the fractional integral (1):

Definition 2.2. *Let $m \leqslant$ Re $\nu < m + 1$, $t > 0$. The operator given by formula*

$$D^{\nu}_t := \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m+1} D^{-(m-\nu+1)}_t f(t) \tag{16}$$

for functions for which the improper integral on the right-hand side of (16) is convergent is called the Riemann–Liouville fractional derivative of order ν.

Let us notice that the functions from the domain of the D_t^ν operator form the subset in the set of functions from definition 2.1. It is a well known fact that this class consists of finite sums of functions of the type

$$t^\lambda \sum_{k=0}^{\infty} a_k t^k \tag{17}$$

or

$$t^\lambda \ln(t) \sum_{k=0}^{\infty} a_k t^k \tag{18}$$

where $\operatorname{Re}\lambda > -1$ and the series have a positive radius of convergence.

In contrast to the fractional integrals the derivative (16) cannot be expressed using only convolution. The formula includes the classical derivative and appears as follows:

$$D_t^\nu f(t) := \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m+1} (f * \Phi_{\nu-m}(t)) \tag{19}$$

with the function $\Phi_{\nu-m} = \frac{t^{-\nu+m}}{\Gamma(m+1-\nu)}$.

We expect the fractional derivative to obey a composition rule analogous to that for a fractional integral. In fact [1, 2] the following formula, which generalizes (3), is valid:

$$D_t^\nu D_t^\mu f = D_t^{\nu+\mu} f \tag{20}$$

provided:

- ν arbitrary, $\mu < \lambda + 1$ and the function f is of the type described by (17,18);
- ν arbitrary, $\mu \geqslant \lambda + 1$ and $a_k = 0$ $k = 0, \ldots, m-1$ for the function f of type (17), (18) where m is the smallest integer greater or equal to $\operatorname{Re}\mu$.

The above formula shows that the fractional derivatives of different orders do not always commute as is the case with the fractional integrals.

The Leibniz rule for the fractional derivative has the form ($\operatorname{Re}\nu \leqslant n-1$) [1, 2, 32–34]

$$D_t^\nu f \cdot g(t) = \sum_{k=0}^{n} \binom{\nu}{k} g^{(k)} \cdot D_t^{\nu-k} f(t) - R_n^\nu(t) \tag{21}$$

when the function f is continuous in the interval $[0, t]$ while g has $n+1$ continuous derivatives in $[0, t]$.

The remainder R_n is the integral expression

$$R_n^\nu(t) = \frac{1}{n!\Gamma(-\nu)} \int_0^t (t-s)^{\nu-1} f(s)\,\mathrm{d}s \int_s^t g^{(n+1)}(\omega)(s-\omega)^n\,\mathrm{d}\omega. \tag{22}$$

If the above remainder goes to 0 for $n \to \infty$ the Leibniz rule (21) can be written for analytic functions in the form of series:

$$D_t^\nu f \cdot g = \sum_{k=0}^{\infty} \binom{\nu}{k} f^{(k)} \cdot D_t^{\nu-k} g. \tag{23}$$

Again the form of Leibniz's rules for the algebra defined by point-wise multiplication of functions is complicated.

We propose to use the algebra of convolution (7). The following statment is valid for the new algebra of functions:

Lemma 2.1. *Let* $m \leqslant \operatorname{Re}\nu < m+1$ *and the function* g *be piecewise continuous in* $(0, +\infty)$. *If the function* f *is a finite sum of functions of the type (17), (18) and fulfils the condition*

$$\lim_{t\to 0+0} f^{(k)} * \Phi_{\nu-m} = 0$$

for $k = 0, 1, \ldots, m$ *then the following rule holds:*

$$D_t^\nu (f * g) = (D_t^\nu f) * g. \tag{24}$$

Proof. We use the well known rule for differentiation of an integral depending on a parameter with the upper limit depending on the same parameter:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t F(t,s)\,\mathrm{d}s = \int_0^t \frac{\partial F(t,s)}{\partial t}\,\mathrm{d}s + \lim_{s\to t-0} F(t,s) \tag{25}$$

and it follows for $0 < \operatorname{Re}\nu < 1$ that

$$D_t^\nu (f * g) = \frac{\mathrm{d}}{\mathrm{d}t}(f * g * \Phi_\nu) = \frac{\mathrm{d}}{\mathrm{d}t}(f * \Phi_\nu * g) = \left(\frac{\mathrm{d}}{\mathrm{d}t}(f * \Phi_\nu)\right) * g = (D_t^\nu f) * g \tag{26}$$

provided

$$\lim_{t\to 0+0} f * \Phi_\nu(t) = 0. \tag{27}$$

Thus when the assumptions are fulfilled formula (24) is valid.

For $m < \operatorname{Re}\nu < m+1$ we apply the rule (25) $m+1$ times:

$$\begin{aligned} D_t^\nu (f * g) &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m+1} (f * g * \Phi_{\nu-m}) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m+1} [(f * \Phi_{\nu-m}) * g] \\ &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m} \left[\left[\frac{\mathrm{d}}{\mathrm{d}t}(f * \Phi_{\nu-m})\right] * g\right] = \cdots = \left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m+1} (f * \Phi_{\nu-m})\right] * g \end{aligned}$$

and arrive at the conditions

$$\lim_{t\to 0+0} f * \Phi_{\nu-m}(t) = 0 \tag{28}$$

$$\lim_{t\to 0+0} f' * \Phi_{\nu-m}(t) = 0 \tag{29}$$

$$\cdots \tag{30}$$

$$\lim_{t\to 0+0} f^{(m)} * \Phi_{\nu-m}(t) = 0 \tag{31}$$

which are fulfilled by assumption. □

The above set of right-sided limits determines the behaviour of the function f in the neighbourhood of $t = 0$, namely $f(t) \sim t^\beta$ with β a complex number fulfilling the condition $\operatorname{Re}\beta > -1 + \operatorname{Re}\nu$.

The symmetric version of formula (24) follows from the commutativity of the Laplace convolution.

Corollary 2.2. *Let* $m \leqslant \operatorname{Re}\nu < m+1$ *and functions* f *and* g *be piecewise continuous in* $(0, +\infty)$. *If both functions* f, g *are finite sums of functions of the type (17), (18) and both of them obey the assumptions from lemma 2.1 then the following rule holds:*

$$D_t^\nu (f * g) = \beta (D_t^\nu f) * g + (1-\beta) f * (D_t^\nu g) \tag{32}$$

for $\beta \in [0, 1]$.

The above lemma together with the composition rule (20) yields the analogue of the property (8) for the Riemann–Liouville fractional derivative:

Corollary 2.3. *Let* $\mathrm{Re}\,\nu > 0$ *and the function* $f * g$ *obey for certain* γ*, fulfilling* $\mathrm{Re}\,\gamma > 0$ *and* $\mathrm{Re}\,(\nu - \gamma) > 0$*, the assumptions of the composition rule (20). If function* f *fulfils the conditions from lemma 2.1 for* γ *and the function* g *the corresponding conditions for* $\nu - \gamma$ *then the following formula holds:*

$$D_t^{\nu}(f * g) = D_t^{\nu-\gamma} D_t^{\gamma}(f * g) = (D_t^{\gamma} f) * D_t^{\nu-\gamma} g. \tag{33}$$

Analogously to (11) we can introduce the new differintegral operator

$$\mathcal{D}_t^{\nu} f(t) := (D_t^{\nu} - 1) f(t). \tag{34}$$

The Leibniz rule for the introduced differintegrable operator of positive order ν is similar to that known from the discrete and noncommutative calculus [29–31]:

$$\mathcal{D}_t^{\nu}(f * g) = (\mathcal{D}_t^{\gamma} f) * g + (\zeta^{\gamma} f) * \mathcal{D}_t^{\nu-\gamma} g \tag{35}$$

$$\mathcal{D}_t^{\nu}(f * g) = (\mathcal{D}_t^{\gamma} f) * \zeta^{\nu-\gamma} g + f * \mathcal{D}_t^{\nu-\gamma} g \tag{36}$$

where we use the notation

$$\zeta^{\gamma} \equiv D_t^{\gamma} \tag{37}$$

and ν and γ together with functions f, g fulfil the conditions from lemma 2.1.

2.3. Riemann–Liouville partial fractional derivatives

Let us extend the formalism introduced in previous sections to the multidimensional case. We shall study the stationarity-conservation equations for some fractional partial differential equations and derive for them the explicit form of stationary currents connected with their symmetries. We assume that in the equation both types of derivative can appear—fractional with respect to to a subset of coordinates and classical—continuous ones with respect to the rest of the coordinates.

Thus the question arises how to define the multiplication of functions. We propose to use the multidimensional Laplace convolution when the initial equation contains only Riemann–Liouville fractional partial derivatives of the form

$$D_k^{\alpha_k} f(\vec{x}) := \frac{1}{\Gamma(m_k + 1 - \alpha_k)} \left(\partial_{x_k}\right)^{m_k+1} \int_0^{x_k} (x_k - s)^{-\alpha_k + m_k} f(\vec{x} + (s - x_k)\vec{e}_k)\, \mathrm{d}s \tag{38}$$

where $m_k \leqslant \mathrm{Re}\,\alpha_k < m_k + 1$. The upper index in the formula denotes the fractional order of the partial derivative while the lower one says that it was taken with respect to coordinate x_k.

Let $x_1, \ldots, x_m$ be a subset of coordinates in our n-dimensional model for which the fractional partial derivatives (38) appear in the equation. Then we define multiplication of functions as follows:

Definition 2.3. *The algebra of functions is defined by the multiplication formula*

$$f * g(\vec{x}) := \int_0^{x_1} \cdots \int_0^{x_m} f\left(\vec{x} - \sum_{l=1}^{m} s_l \vec{e}_l\right) g\left(\vec{x} + \sum_{l=1}^{m} (s_l - x_l)\vec{e}_l\right) \mathrm{d}s_1 \ldots \mathrm{d}s_m \tag{39}$$

where $(\vec{e}_l)_k = \delta_{lk}$.

Similarly to the one-dimensional case the multiplication (39) is associative and commutative.

In the above algebra of functions the Leibniz rule (32) given by corollary 2.2 is valid for functions fulfilling the respective assumptions concerning their behaviour at $x_k = 0$:

$$D_k^{\alpha_k} f * g = \beta_k (D_k^{\alpha_k} f) * g + (1 - \beta_k) f * D_k^{\alpha_k} g \tag{40}$$

with $\beta_k \in [0, 1]$ for $k = 1, \ldots, m$.

For classical derivatives acting by assumption in directions $j = m + 1, \ldots, n$ we obtain for convolution (39) the standard form of the Leibniz rule

$$\partial_j (f * g) = (\partial_j f) * g + f * \partial_j g. \tag{41}$$

Similarly to the one-dimensional case investigated in the previous section we can introduce also the partial differintegral operators of positive order for functions fulfilling suitable conditions:

$$\mathcal{D}_k^{\alpha_k} f(\vec{x}) := D_k^{\alpha_k} f(\vec{x}) - f(\vec{x}). \tag{42}$$

These operators obey the Leibniz rule for functions multiplied according to (39):

$$\mathcal{D}_k^{\alpha_k}(f * g) = (\mathcal{D}_k^{\gamma_k} f) * g + (\zeta_k^{\gamma_k} f) * \mathcal{D}_k^{\alpha_k - \gamma_k} g \tag{43}$$

$$\mathcal{D}_k^{\alpha_k}(f * g) = (\mathcal{D}_k^{\gamma_k} f) * \zeta_k^{\alpha - \gamma_k} g + f * \mathcal{D}_k^{\alpha_k - \gamma_k} g \tag{44}$$

$$\partial_j (f * g) = (\partial_j f) * g + f * \partial_j g \tag{45}$$

where $k = 1, \ldots, m$ and $j = m+1, \ldots, n$ and the function f obeys the conditions of lemma 2.1 for the fractional order of the derivative γ_k while the second function g respectively fulfils these conditions for $\alpha_k - \gamma_k$.

The first two formulae are the symmetric forms of the Leibniz rule for fractional derivatives and the last one is the standard Leibniz rule for partial derivatives but taken in the algebra of functions defined by multiplication (39).

Now we can apply the properties of multiplication (39) and fractional differentation in construction of the stationarity-conservation laws and conserved charges for some partial fractional equations.

3. Examples

3.1. Fractional diffusion equation in 1 + 1

Let us recall the fractional diffusion equation discussed in [3, 12, 14]:

$$D_t^{\alpha} \phi(x,t) = \lambda^2 \partial_x^2 \phi(x,t) + \phi(x,0) \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \tag{46}$$

where $t > 0$, $x \in R$ and $0 < \alpha < 1$ describes the process of ultraslow diffusion while the value $1 < \alpha < 2$ is used for intermediate processes [3, 12, 14].

Let us focus on the case of ultraslow diffusion. The operator of the equation contains both types of derivative, fractional with respect to time and standard for the spatial dimension:

$$\Lambda(D_t^{\alpha}, \partial_x) = D_t^{\alpha} - \lambda^2 \partial_x^2. \tag{47}$$

The product of functions for this model is defined according to (39) and appears as follows:

$$f * g(x,t) = \int_0^t f(x, t-s) g(x,s)\, \mathrm{d}s. \tag{48}$$

Using the properties of the new multiplication and of the fractional derivative (32) we construct the operator Γ with components

$$\Gamma_x = \lambda^2 \overleftarrow{\partial}_x - \lambda^2 \partial_x \qquad \Gamma_t = 2. \tag{49}$$

Then the current

$$J_x = \phi' \Gamma_x * \phi = \phi' \lambda^2 \overleftarrow{\partial}_x * \phi - \phi' * \lambda^2 \partial_x \phi \tag{50}$$

$$J_t = \phi' \Gamma_t * \phi = 2\phi' * \phi \tag{51}$$

obeys the stationarity-conservation equation for $t \geqslant 0$ in the area $\phi(x,0) = \phi'(x,0) = 0$

$$\partial_x J_x + D_t^\alpha J_t = 0 \tag{52}$$

provided the function ϕ is the solution of initial equation (46) while ϕ' solves its conjugation:

$$\Lambda(-D_t^\alpha, -\partial_x)\phi'(x,t) + \phi'(x,0)\frac{t^{-\alpha}}{\Gamma(1-\alpha)} = 0. \tag{53}$$

Before passing to the proof of the stationarity-conservation law (52) we shall discuss the existence of solutions of the diffusion equation and of its conjugated form with required properties around $t = 0$.

Let us recall the form of the general solution of equation (46) [14]:

$$\phi(x,t) = \int_{-\infty}^{\infty} \mathrm{d}y\, G_\alpha(x,y,t)\phi(y,0) \tag{54}$$

where G_α is the fractional Green function of the following form:

$$G_\alpha(x,y,t) = t^{-\alpha}\int_0^\infty \mathrm{d}z\, E_\alpha(t^{-\alpha}z)G(x,y,z) = \int_0^\infty \mathrm{d}v\, E_\alpha(v)G(x,y,t^\alpha v) \tag{55}$$

with the function $G(x,y,z) = G(|x-y|,z)$ being the standard Green function

$$G(|x-y|,z) = \frac{1}{\sqrt{4\pi z}}\mathrm{e}^{-\frac{|x-y|^2}{4z}}$$

and E_α denoting the Mittag–Leffler function [2, 14].

Taking into account the asymptotic properties of the function G we conclude that the solution ϕ behaves in the neighbourhood of $t = 0$ as the power function $t^{-\frac{\alpha}{2}}$. The solution ϕ' of the conjugated equation has a similar form so its behaviour for $t \to 0$ is the same as that of the considered solution ϕ.

This fact implies that at least for $0 < \alpha < \frac{2}{3}$ the assumptions of lemma 2.1 are fulfilled; therefore, we can use in the proof of the stationarity-conservation equation (52) the Leibniz rule for the fractional derivative D_t^α given in formula (32).

Let us check the conservation law explicitly applying the Leibniz rule (32) with $\beta = \frac{1}{2}$:

$$\begin{aligned}\partial_x J_x + D_t^\alpha J_t &= \partial_x\big(\phi'\lambda^2 \overleftarrow{\partial}_x * \phi - \phi' * \lambda^2\partial_x\phi\big) + D_t^\alpha\,\big(2\phi' * \phi\big)\\ &= \lambda^2(\partial_x^2\phi') * \phi - \phi' * \lambda^2\partial_x^2\phi - (-D_t^\alpha\phi') * \phi + \phi' * D_t^\alpha\phi\\ &\quad -\big[\big(-D_t^\alpha - \lambda^2\partial_x^2\big)\phi'\big] * \phi + \phi' * \big(D_t^\alpha - \lambda^2\partial_x^2\big)\phi = 0.\end{aligned} \tag{56}$$

We have omitted the terms depending on initial values $\phi(x,0)$ and $\phi'(x,0)$ as we expect the rule (52) to be fulfilled in the area where $\phi(x,0) = \phi'(x,0) = 0$.

Having obtained the general form of stationary current (50) and (51) we can discuss the possible symmetries of equation (46), which can be used in construction of different solutions of the initial diffusion problem. The set includes spatial momentum $P_x = \partial_x$ as this operator commutes with the operator of diffusion equation (46).

The stationarity-conservation laws for currents including new solutions are fulfilled for the transformed solution $P_x\phi$ in the area $\partial_x\phi(x,0) = \phi'(x,0) = 0$. The simplest possible choice of initial value for solutions of the diffusion equation and of its conjugation is $\frac{\phi(x,0)}{\phi_0} = \frac{\phi'(x,0)}{\phi_0'} = \delta(x)$ with ϕ_0 and ϕ_0' arbitary constants.

In this way we arrive at the stationary (for $x \neq 0$ in the sense of (52)) current connected with symmetry of the fractional diffusion equation:

$$J_x^x = \phi'\Gamma_x * P_x\phi \qquad J_t^x = \phi'\Gamma_t * P_x\phi. \tag{57}$$

The stationarity-conservation equation (52) can be reformulated using the definition of the Riemann–Liouville derivative so as to obtain the standard conservation equation, namely

$$\partial_x J'_x + \partial_t J'_t = 0 \tag{58}$$

which is fulfilled for $x \neq 0$, and the components of the new current appear as follows:

$$J'_x = J_x \qquad J'_t = J_t * \Phi_\alpha = \frac{1}{\Gamma(1-\alpha)} J_t * t^{-\alpha}. \tag{59}$$

Following the classical field theory the time components of the conserved currents J and J' yield the charges

$$Q = \int_{-\infty}^{\infty} \mathrm{d}x \quad J_t \tag{60}$$

$$Q' = \int_{-\infty}^{\infty} \mathrm{d}x \quad J'_t. \tag{61}$$

The respective derivatives of the above charges are determined by the boundary terms for time components of the currents and the initial conditions for solutions ϕ and ϕ':

$$D_t^\alpha Q = -\lim_{x\to\infty} \left[\lambda^2(\partial_x\phi') * \phi - \lambda^2\phi' * \partial_x\phi\right] + \lim_{x\to-\infty} \left[\lambda^2(\partial_x\phi') * \phi - \lambda^2\phi' * \partial_x\phi\right] + \phi'_0 \frac{t^{-\alpha}}{\Gamma(1-\alpha)} * \phi(0,t) + \phi'(0,t) * \phi_0 \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \tag{62}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} Q' = D_t^\alpha Q. \tag{63}$$

From the general form of solutions (54) we obtain for initial condition $\phi(x, t = 0) = \phi_0\delta(x)$

$$\phi(0,t) = \phi_0 G_\alpha(0,0,t)$$

and for the conjugated equation

$$\phi'(0,t) = -\phi'_0 G_\alpha(0,0,t).$$

Due to this property of the solutions and the commutativity of the convolution the last terms in the above formulae cancel. The first parts vanish by the asymptotic properties of the Green function, which decays exponentially together with its spatial derivative for large x.

Thus the explicit expressions for charges (60) and (61) produce the stationary function Q and constant function Q' connected with the stationarity law and conservation law of the diffusion equation in 1 + 1 dimensions:

$$D_t^\alpha Q = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}t} Q' = 0. \tag{64}$$

3.2. Generalized fractional diffusion

Let us now extend the dimension of the spacelike coordinates to d. We shall consider the equation known as the generalized fractional diffusion problem [11, 35]:

$$D_t^\alpha \phi(\vec{x}, t) = C\triangle\phi(\vec{x}, t) + \phi(\vec{x}, 0)\frac{t^{-\alpha}}{\Gamma(1-\alpha)} \tag{65}$$

with $t > 0$, $0 < \alpha < 1$ and $\triangle$ the Laplace operator in d-dimensional Euclidean space.

We again check the properties in the neighbourhood of $t = 0$ of the solutions.

The solution for arbitrary initial condition generalizes formula (54) used in the previous section:

$$\phi(\vec{x}, t) = \int_{-\infty}^{\infty} \mathrm{d}^d\, \vec{y} G_\alpha(\vec{x}, \vec{y}, t)\phi(\vec{y}, 0) \tag{66}$$

where G_α is the fractional Green function of the following form:

$$G_\alpha(\vec{x}, \vec{y}, t) = t^{-\alpha} \int_0^\infty \mathrm{d}z\, E_\alpha(t^{-\alpha} z) G(\vec{x}, \vec{y}, z) \tag{67}$$

with the function $G(\vec{x}, \vec{y}, z) = G(|\vec{x} - \vec{y}|, z)$ being the standard Green function

$$G(|\vec{x} - \vec{y}|, z) = (4\pi z)^{-\frac{d}{2}} \mathrm{e}^{-\frac{|\vec{x}-\vec{y}|^2}{4z}}$$

and E_α denoting the Mittag–Leffler function.

Taking into account the fact that this function for $0 < \alpha < 1$ is an entire function and vanishes exponentially for large positive values of argument we conclude that the solution ϕ behaves in the neighbourhood of $t = 0$ as the power function $t^{-\alpha}$. A similar argument applies to the solution of the conjugated equation ϕ' given below (72).

The product of functions given by (39) has in the $(d + 1)$-dimensional case the following explicit form:

$$f * g(\vec{x}, t) = \int_0^t f(\vec{x}, t - s) g(\vec{x}, s)\, \mathrm{d}s. \tag{68}$$

The number of components of the operator Γ and of the current J is now $d + 1$ while the form of the spacelike and timelike parts is identical to those obtained for the modified Nigmatullin diffusion equation.

The operator Γ given by

$$\Gamma_i = C \overleftarrow{\partial}_{x_i} - C \partial_{x_i} \qquad \Gamma_t = 2 \tag{69}$$

can be applied in the construction of the current

$$J_i = \phi' \Gamma_i * \phi = \phi' C \overleftarrow{\partial}_{x_i} * \phi - \phi' * C \partial_{x_i} \phi \tag{70}$$

$$J_t = \phi' \Gamma_t * \phi = 2\phi' * \phi \tag{71}$$

where ϕ solves the initial generalized diffusion equation (65) and ϕ' its conjugation:

$$\Lambda(-D_t^\alpha, -\partial_{x_1}, \ldots, -\partial_{x_d}) \phi'(\vec{x}, t) + \phi'(\vec{x}, 0) \frac{t^{-\alpha}}{\Gamma(1 - \alpha)} = 0. \tag{72}$$

The current (70), (71) obeys the stationarity-conservation equation:

$$\sum_{i=1}^{d} \partial_{x_i} J_i + D_t^\alpha J_t = 0 \tag{73}$$

for $\vec{x} \neq \vec{0}$ provided the solution of diffusion equation (65) with the initial condition $\phi(\vec{x}, 0) = \phi_0 \delta(\vec{x})$ is taken and for the conjugated equation the initial condition $\phi'(\vec{x}, 0) = \phi'_0 \delta(\vec{x})$ is considered.

The proof of the above conservation law is analogous to that presented in the previous section for the 1 + 1 diffusion equation. The essential features in the proof are the asymptotic properties of the solutions ϕ and ϕ' in the neighbourhood of $t = 0$. Similarly to the previous case they allow us to apply the Leibniz rule (32) for $\vec{x} \neq \vec{0}$ at least when $0 < \alpha < \frac{1}{2}$.

The set of symmetry operators for the $d + 1$ case is much wider as it contains not only momenta

$$P_i = \partial_{x_i} \tag{74}$$

but also the angular momentum with respect to the spacelike dimensions

$$M_{ij} = x_i \partial_{x_j} - x_j \partial_{x_i} \tag{75}$$

where $i, j = 1, \ldots, d$.

As the symmetry operators transform solutions of (65) into solutions with the same properties around $t = 0$ we can use them in construction of the stationary-conserved currents:

$$J_i^\delta = \phi' \Gamma_i * \delta\phi = \phi' C \overleftarrow{\partial}_{x_i} * \delta\phi - \phi' * C \partial_{x_i} \delta\phi \tag{76}$$

$$J_t^\delta = \phi' \Gamma_t * \delta\phi = 2\phi' * \delta\phi \tag{77}$$

where δ denotes one of the above symmetry operators of equation (65).

The currents (76) and (77) can be transformed into components J' similarly as in the case of the 1 + 1 fractional diffusion. Taking the components

$$J_i'^\delta = J_i^\delta \qquad J_t'^\delta = J_t^\delta * \Phi_\alpha = \frac{1}{\Gamma(1-\alpha)} J_t^\delta * t^{-\alpha} \tag{78}$$

we obtain the conservation law for the $d + 1$ fractional diffusion process:

$$\sum_{i=1}^{d} \partial_{x_i} J_i'^\delta + \partial_t J_t'^\delta = 0 \tag{79}$$

valid for $\vec{x} \neq \vec{0}$.

Finally the derived set of stationary and conserved currents yields two sets of charges indexed by the symmetry operators $\delta \in \{P_i, M_{ij}\}$ $i, j = 1, \ldots, d$:

$$Q^\delta = \int_\infty^\infty \mathrm{d}^d\vec{x} \quad J_t^\delta \tag{80}$$

$$Q'^\delta = \int_\infty^\infty \mathrm{d}^d\vec{x} \quad J_t'^\delta \tag{81}$$

which are respectively stationary and conserved functions of time:

$$D_t^\alpha Q^\delta = 0 \tag{82}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} Q'^\delta = 0. \tag{83}$$

4. Stationarity-conservation laws for some fractional partial equations

In previous sections we have obtained the stationarity law and conservation law for some examples of partial fractional differential equations.

In the multidimensional case of diffusion equation the general solution allows the explicit construction of the current which obeys the stationarity-conservation law in the area of space where the initial conditions vanish both for solution of the diffusion equation and for its conjugation.

We have shown that the stationary currents yield stationary charges which can be converted to conserved ones.

The discussed example shows that the construction of possible charges, stationary or conserved, is connected with the asymptotic properties of solutions around 0 and for $|\vec{x}| \to \infty$.

In the following we shall discuss the general construction of stationarity-conservation laws assuming that regular (in the sense of lemma 2.1) solutions of the respective fractional differential equations exist at least in a certain area of space.

4.1. Mixed fractional differential and differential partial equations

Let us consider now the general equation which contains the fractional and differential parts of the following form:

$$\Lambda(D, \partial)\phi = [\tilde{\Lambda}(D) + \Lambda(\partial)]\phi = \left(\sum_{k=1}^{m} \tilde{\Lambda}_k D_k^{\alpha_k} + \sum_{l=1}^{N} \Lambda_{\mu_1 \ldots \mu_l} \partial^{\mu_1} \ldots \partial^{\mu_l} + \Lambda_0 \right) \phi = 0. \tag{84}$$

The introduced equation has constant coefficients (we admit also constant matrices) and generalizes the studied examples of fractional diffusion. We shall study the construction for the homogenous form of the equation, remembering that the addition of the initial terms similar to those discussed previously restricts only the area of application of the stationarity equation and does not change the general construction. We assume that for given variables $x_1, \ldots, x_m$ the equation includes only fractional derivatives in $\tilde{\Lambda}(D)$, while for the remaining coordinates $x_{m+1}, \ldots, x_n$ only partial derivatives appear in the operator $\Lambda(\partial)$.

To derive the stationarity-conservation law we shall use the Takahashi–Umezawa method [28] for the differential part $\Lambda(\partial)$ and the fractional Leibniz rule (32) for the part $\tilde{\Lambda}(D)$ containing fractional operators.

As we know from the discussed examples each direction of space yields the component of the current which for coordinates $x_1, \ldots, x_k$ is given by the $\tilde{\Gamma}$ operator of the form

$$\tilde{\Gamma}_k = 2\tilde{\Lambda}_k \tag{85}$$

while for the part $j = m+1, \ldots, n$ we obtain [28]

$$\Gamma_j = \sum_{l=1}^{N-1} \sum_{k=1}^{l} \Lambda_{j\mu_1\ldots\mu_l} (-\overleftarrow{\partial}^{\mu_1}) \ldots (-\overleftarrow{\partial}^{\mu_k}) \partial^{\mu_{k+1}} \ldots \partial^{\mu_l}. \tag{86}$$

It is a well known fact that for an arbitrary pair of functions f and g the operator Γ fulfils the equality

$$\sum_{j=m+1}^{n} \partial^j f * \Gamma_j g = -f\Lambda(-\overleftarrow{\partial}) * g + f * \Lambda(\partial) g \tag{87}$$

where the multiplication is given by the convolution (39) and $\Lambda(-\overleftarrow{\partial})$ is the conjugated operator for $\Lambda(\partial)$ acting on the left-hand side.

The above property of the Γ operator together with the Leibniz rule (32) for fractional derivatives (taken with parameters $\beta_k = \frac{1}{2}$ $\quad k = 1, \ldots, m$) implies the following proposition to be valid:

Proposition 4.1. *Let the function ϕ be an arbitrary solution of the equation (84) and let ϕ' solve the conjugated equation:*

$$\begin{aligned} \phi'\Lambda(-\overleftarrow{D}, -\overleftarrow{\partial}) &= \phi'[\tilde{\Lambda}(-\overleftarrow{D}) + \Lambda(-\overleftarrow{\partial})] \\ &= \phi'\left(-\sum_{k=1}^{m} \tilde{\Lambda}_k \overleftarrow{D}_k^{\alpha_k} + \sum_{l=1}^{N} \Lambda_{\mu_1\ldots\mu_l} (-\overleftarrow{\partial}^{\mu_1}) \ldots (-\overleftarrow{\partial}^{\mu_l}) + \Lambda_0 \right) = 0. \end{aligned} \tag{88}$$

Then the current given by the components

$$J_k = \phi' * \tilde{\Lambda}_k \phi + \phi' \tilde{\Lambda}_k * \phi \qquad k = 1, \ldots, m \tag{89}$$

$$J_j = \phi' * \Gamma_j \phi \qquad j = m+1, \ldots, n \tag{90}$$

fulfils the stationarity-conservation equation

$$\sum_{k=1}^{m} D_k^{\alpha_k} J_k + \sum_{j=m+1}^{n} \partial^j J_j = 0 \tag{91}$$

provided the solutions ϕ and ϕ' fulfil the conditions of lemma 2.1 in the neighbourhood of $x_k = 0$ $\quad k = 1, \ldots, m$.

Proof. We check the law (91) explicitly:

$$\sum_{k=1}^{m} D_k^{\alpha_k} J_k + \sum_{j=m+1}^{n} \partial^j J_j = \sum_{k=1}^{m} D_k^{\alpha_k} \left(\phi' * \tilde{\Lambda}_k \phi + \phi' \tilde{\Lambda}_k * \phi\right) + \sum_{j=m+1}^{n} \partial^j \left(\phi' * \Gamma_j \phi\right)$$

$$= \sum_{k=1}^{m} (D_k^{\alpha_k} \phi') \tilde{\Lambda}_k * \phi + \sum_{k=1}^{m} \phi' * \tilde{\Lambda}_k D_k^{\alpha_k} \phi - \phi' \Lambda(-\overleftarrow{\partial}) * \phi + \phi' * \Lambda(\partial)\phi$$

$$= -\phi' \Lambda(-\overleftarrow{D}, -\overleftarrow{\partial}) * \phi + \phi' * \Lambda(D, \partial)\phi = 0.$$

Thus for every equation of the form (84) we can produce the exact form of the stationary-conserved current provided the initial equation and its conjugation have solutions which fulfil the asymptotic conditions at $x_k = 0 \quad k = 1, \ldots, m$ and which allow the application of the Leibniz rule for fractional partial derivatives $D_k^{\alpha_k}$. □

The stationarity-conservation equation (91) can be rewritten in the form of the standard conservation law for modified components of the above current ($m_k < \alpha_k < m_k + 1 \quad k = 1, \ldots, m$):

$$J_k' = (\partial^k)^{m_k} (J_k *_k \Phi_{\alpha_k - m_k}) \qquad k = 1, \ldots, m \tag{92}$$

$$J_j' = J_j \qquad j = m+1, \ldots, n \tag{93}$$

where the convolution $*_k$ is given by the formula

$$f *_k g(\vec{x}) = \int_0^{x_k} f(\vec{x} - s\vec{e}_k) g(\vec{x} + (s - x_k)\vec{e}_k)\, \mathrm{d}s_k. \tag{94}$$

The new current J' obeys the conservation law

$$\sum_{l=1}^{n} \partial^l J_l' = 0. \tag{95}$$

4.2. Mixed fractional sequential and differential partial equations

In the previous construction we have considered the fractional part of the operator including only the first power of the corresponding partial fractional derivatives while in the differential part we have taken an arbitrary polynomial of partial derivatives. Let us extend the derivation of the stationarity-conservation laws to the general case containing both a polynomial of fractional derivatives and a polynomial of classical partial derivatives

$$\Lambda(D, \partial)\phi = [\tilde{\Lambda}(D) + \Lambda(\partial)]\phi$$

$$= \left(\sum_{k=1}^{M} \tilde{\Lambda}_{\rho_1 \ldots \rho_k} D_{\rho_1}^{\alpha_1} \ldots D_{\rho_k}^{\alpha_k} + \sum_{l=1}^{N} \Lambda_{\mu_1 \ldots \mu_l} \partial^{\mu_1} \ldots \partial^{\mu_l} + \Lambda_0 \right) \phi = 0. \tag{96}$$

The derivatives with respect to the coordinates $x_1, \ldots, x_m$ are the fractional $D_{\rho_i}^{\alpha_i}$ where the upper index denotes the fractional order and the lower one the respective partial direction. The part depending on fractional derivatives has now the form of a partial sequential fractional operator generalizing the sequential operator for one-dimensional space [2]. The coefficients Λ and $\tilde{\Lambda}$ are again constant matrices or numbers. As the derivatives with respect to different coordinates do commute both types of coefficient are fully symmetric with respect to the permutation of the set of indices.

To obtain the Γ operator fulfilling the equation (87) we again use the Takahashi–Umezawa method for the differential part $\Lambda(\partial)$ and obtain the components Γ_j as given by (86) whereas for $\tilde{\Gamma}$ we have

$$\tilde{\Gamma}_k = 2 \sum_{j=1}^{M-1} \sum_{l=1}^{j} \tilde{\Lambda}_{k\rho_1 \ldots \rho_j} (-\overleftarrow{D}_{\rho_1}^{\alpha_1}) \ldots (-\overleftarrow{D}_{\rho_l}^{\alpha_l}) D_{\rho_{l+1}}^{\alpha_{l+1}} \ldots D_{\rho_j}^{\alpha_j}. \tag{97}$$

It is easy to check the analogue of the formula (87) for the operator $\tilde{\Gamma}$:

$$\sum_{k=1}^{m} D_k^{\alpha_k}\big(f * \tilde{\Gamma}_k g\big) = -f\tilde{\Lambda}(-\overleftarrow{D}) * g + f * \tilde{\Lambda}(D)g \tag{98}$$

for an arbitrary pair of functions f and g allowing the use of the Leibniz rule (32) together with their fractional derivatives $D_{\rho_{l+1}}^{\alpha_{l+1}} \dots D_{\rho_j}^{\alpha_j} g$ and $f(-\overleftarrow{D}_{\rho_1}^{\alpha_1}) \dots (-\overleftarrow{D}_{\rho_l}^{\alpha_l})$.

All the above calculations yield as a result the following proposition, which describes the explicit construction of the stationarity-conservation law for a linear sequential fractional–differential equation (96):

Proposition 4.2. *Let the function ϕ be an arbitrary solution of equation (96) and let ϕ' be a solution of the conjugated equation in the form*

$$\begin{aligned} 0 &= \phi'\Lambda(-\overleftarrow{D}, -\overleftarrow{\partial}) \\ &= \phi'\left(\sum_{k=1}^{M} \tilde{\Lambda}_{\mu_1\dots\mu_k}(-\overleftarrow{D}_{\mu_1}^{\alpha_1}) \dots (-\overleftarrow{D}_{\mu_k}^{\alpha_k}) + \sum_{l=1}^{N} \Lambda_{\mu_1\dots\mu_l}(-\overleftarrow{\partial}^{\mu_1}) \dots (-\overleftarrow{\partial}^{\mu_l}) + \Lambda_0\right). \end{aligned} \tag{99}$$

Then the current with the components

$$J_k = \phi' * \tilde{\Gamma}_k\phi \qquad k = 1, \dots, m \tag{100}$$

$$J_j = \phi' * \Gamma_j\phi \qquad j = m+1, \dots, n \tag{101}$$

obeys the stationarity-conservation equation

$$\sum_{k=1}^{m} D_k^{\alpha_k} J_k + \sum_{j=m+1}^{n} \partial^j J_j = 0 \tag{102}$$

provided the solutions ϕ and ϕ' together with their derivatives appearing in the formulae for components (100) fulfil the conditions of lemma 2.1 in the neighbourhood of $x_k = 0 \quad k = 1, \dots, m$.

Proof. We use the properties of the solutions and of the operators Γ and $\tilde{\Gamma}$ and obtain

$$\sum_{j=m+1}^{n} \partial^j J_j = \sum_{j=m+1}^{n} \partial^j(\phi' * \Gamma_j\phi) = -\phi'\Lambda(-\overleftarrow{\partial}) * \phi + \phi' * \Lambda(\partial)\phi$$

$$\sum_{k=1}^{m} D_k^{\alpha_k} J_k = \sum_{k=1}^{m} D_k^{\alpha_k}(\phi' * \tilde{\Gamma}_k\phi) = -\phi'\tilde{\Lambda}(-\overleftarrow{D}) * \phi + \phi' * \tilde{\Lambda}(D)\phi.$$

Thus the left-hand side of the stationarity-conservation formula is of the form

$$\begin{aligned} &\sum_{k=1}^{m} D_k^{\alpha_k} J_k + \sum_{j=m+1}^{n} \partial^j J_j \\ &\quad = -\phi'\big(\tilde{\Lambda}(-\overleftarrow{D}) + \Lambda(-\overleftarrow{\partial}) + \Lambda_0\big) * \phi + \phi' * \big(\tilde{\Lambda}(D) + \Lambda(\partial) + \Lambda_0\big)\phi = 0 \end{aligned}$$

and vanishes on shell. □

We can rewrite the stationarity-conservation law to have the conservation law connected with equation (96). To this end we apply the definition of the Riemann–Liouville fractional derivative (38). The modified components of the current have the form similar to that derived in the previous section ($m_k < \alpha_k < m_k + 1 \quad k = 1, \dots, m$):

$$J'_k = (\partial^k)^{m_k}\left(J_k *_k \Phi_{\alpha_k - m_k}\right) \qquad k = 1, \dots, m \tag{103}$$

$$J'_j = J_j \qquad j = m+1, \dots, n \tag{104}$$

with the convolution $*_k$ given by (94).

They obey the conservation law

$$\sum_{l=1}^{n} \partial^l J_l' = 0. \qquad (105)$$

4.3. Stationary and conserved charges for mixed fractional–differential models

Following the results obtained for fractional diffusion we shall apply the derived stationarity-conservation law in construction of stationary charges.

Two cases should be considered: when the time derivative in the operator of the equation is a fractional and when it is a standard partial one.

Let us assume that the time derivative in equations (84), (96) is a fractional one. Integrating the time component of the current fulfilling the stationarity-conservation equations (91), (102) we arrive at the charge

$$Q = \int_{R^{n-1}} \mathrm{d}\vec{x} \quad J_t(\vec{x}, t) \qquad (106)$$

which is a stationary function of order α_t, which also determines the order of the fractional time derivative

$$D_t^{\alpha_t} Q = 0 \qquad (107)$$

provided the respective boundary terms vanish. For components $J_j \quad j = m+1, \ldots, n$ this means that they vanish at infinity in the given directions while for components $J_k \quad k = 2, \ldots, m$ the asymptotic condition has the form

$$\lim_{|x_k| \to \infty} (\partial^k)^{m_k} (J_k *_k \Phi_{\alpha_k - m_k}) = 0 \qquad (108)$$

where $m_k < \alpha_k < m_k + 1$.

The second possibility is the model with standard time derivative. Then the charge

$$Q = \int_{R^{n-1}} \mathrm{d}\vec{x} \quad J_t(\vec{x}, t) \qquad (109)$$

is a strictly stationary function of time, that means it is a true constant function:

$$\partial^t Q = 0 \qquad (110)$$

when the asymptotic conditions for respective components of the currents are fulfilled:

$$\lim_{|x_j| \to \infty} J_j = 0 \qquad j = m+2, \ldots, n \qquad (111)$$

$$\lim_{|x_k| \to \infty} (\partial^k)^{m_k} (J_k *_k \Phi_{\alpha_k - m_k}) = 0 \qquad k = 1, \ldots, m. \qquad (112)$$

The exact form of the symmetry algebra of the equations (84) and (96) varies for different examples. Let us however notice that it includes for all of them the momenta:

$$P_k = D_k^{\alpha_k} \qquad k = 1, \ldots, m \qquad (113)$$

$$P_j = \partial^j \qquad j = m+1, \ldots, n \qquad (114)$$

as they commute with the operator of these equations.

However if we propose to use the above momenta in derivation of conserved currents and charges we must additionally assume the regular behaviour of the $W(D) P_k \phi$ and $W(D) P_j \phi$ functions in the neighbourhood of zero with respect to the $x_1, \ldots, x_m$ coordinates ($W(D)$ denote the polynomials of fractional derivatives appearing in the formula for the $\tilde{\Gamma}$ operator).

When this assumption is fulfilled the stationary-conserved currents appear as follows:

$$J_k^\delta = \phi' * \tilde{\Gamma}_k \delta\phi \qquad k = 1, \ldots, m \tag{115}$$

$$J_j^\delta = \phi' * \Gamma_j \delta\phi \qquad j = m+1, \ldots, n \tag{116}$$

where the operators $\tilde{\Gamma}$ and Γ are given explicitly in previous sections.

In this case we have the family of stationary (or respectively conserved charges) depending which of the two cases considered apply to our model. They have the following explicit form:

$$Q^\delta = \int_{R^{n-1}} \mathrm{d}\vec{x}\, \phi' * \tilde{\Gamma}_t \delta\phi \tag{117}$$

for the case where the time derivative is fractional and for the standard time derivative we have

$$Q^\delta = \int_{R^{n-1}} \mathrm{d}\vec{x}\, \phi' * \Gamma_t \delta\phi \tag{118}$$

where δ is one of the momentum operators given in (113) and (114).

5. Conclusions

We have discussed the Leibniz rule for the algebra of Laplace convolution of differintegrable functions.

The derived procedure for construction of the nonlocal stationary currents applies to fractional differential linear equations including Riemann–Liouville fractional and classical derivatives provided there exist the regular solutions of the initial and conjugated equations. A similar method is being investigated also for Weyl fractional derivatives for the algebra of functions defined by Fourier convolution. It seems that it can be extended to models with fractional derivatives defined via a generalized function approach as well.

For the general case we have extracted the explicit form of stationary-conserved currents assuming the regularity of solutions in the neighbourhood of 0. It was shown that the stationary currents are connected with the conserved ones. Both types of current produce charges: namely stationary currents yield stationary charges and respectively from conserved nonlocal currents we obtain integrals of motion.

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